Displacement Discontinuity Analysis of the Effects of Various Hydraulic Fracturing Parameters on the Crack Opening Displacement (COD)

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ABSTRACT

The combination of horizontal drilling along with hydraulic fracturing has significantly improved the production of hydrocarbon reservoirs and made it possible to extract the relatively impermeable and uneconomical reservoirs. The production rate of oil and gas wells increases proportional to hydraulic fracture aperture or crack opening displacement (COD). This is an important parameter in fracture mechanics literature and hydraulic fracturing of hydrocarbon reservoirs. Despite the significance of COD there are a few analytical solutions for the estimation of COD under certain conditions. In this paper the effect of various parameters on COD is investigated semi-analytically. A higher order displacement discontinuity method is used to consider the effects of different parameters (Young's modulus, Poisson's ratio, internal pressure, maximum and minimum horizontal stresses, crack half-length and its inclination with maximum horizontal stress) on the COD in a hydraulic fracturing process under arbitrarily conditions. The coefficient of determination and standard error of the estimate were 94.35% and 4.37×10^{-4} respectively, showing a good agreement between the fitted equation and the numerical results. The effect of propagation and well radius on the maximum COD was also investigated. The results showed that COD increases almost linearly with the crack propagation and increase of well radius of hydraulic fractures (HFs). These effects are more significant when HFs are propagating in the direction of maximum horizontal stress. The proposed equation and the results from propagation of hydraulic fractures can be used in early stages of a hydraulic fracturing design.

Keywords: Crack opening displacement, hydraulic fracturing, higher order displacement discontinuity method, crack propagation.

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INTRODUCTION

Application of horizontal drilling in conjunction with the hydraulic fracturing process has significantly improved the production of hydrocarbon reservoirs so that the extraction from otherwise uneconomical reservoirs is being possible today. Therefore, the propagation of hydraulic fractures has been the focus of many studies [1–5]. At least 6 important advantages will be achieved through horizontal drilling which includes:

- Reduction of water and gas coning.
- Increase of production rate
- Lower pressure drop around the wellbore
- Better sand production control
- Larger drainage coefficient pattern leading to higher recovery rate

Most studies in this field utilize linear elastic fracture mechanics (LEFM) principles to simulate the fracture propagation process. Stress intensity factor (SIF) and energy release rate form the foundation of LEFM theory. They are used to predict the fracture of a cracked brittle solid (such as rock) based on LEFM concept. In the theory of LEFM, the nonlinear effects are assumed to be minimal and negligible, and the cracked solid is treated as a linearly elastic medium [4]. However, the SIF, along with the energy release rate approach, would become futile under large-scale plastic deformations [3]. Several fracture parameters have been proposed to predict fracture of solids under nonlinear deformation conditions, such as the J integral which is a generalization of the energy release rate concept to nonlinear elastic materials by Rice [6], the crack opening displacement (COD) and the crack tip opening displacement (CTOD) by Wells [7,8], and the crack tip opening angle (CTOA). Wells in sequence of his two papers [7,8] proposed

the use of COD as a fracture parameter. The COD criterion is equivalent to the effective stress intensity factor criterion under modest yielding conditions but can be extended to the large-scale yielding conditions when it is combined with the COD equation based on Dugdale model [8]. Aperture of a hydraulic fracture (HF) or COD plays an important role in the production of hydrocarbon reservoirs. The higher the value of COD is achieved in HFs; the higher production rate will be. However, most studies have been focused on HF propagation path and the parameters affecting the amount of COD have not been fully studied. While there are many analytically and numerically derived equations to estimate the SIFs and predict crack propagation paths and directions, there are not much of such equations in the case of COD prediction. The most well-known analytical solution for crack opening displacement proposed by Sneddon which only considers a horizontal crack under uniform internal pressure and far-field stress acting perpendicular to the crack faces. Once the crack rotates, or the far-field stress or stresses are not perpendicular to the crack faces, the Sneddon's analytical solution loses its applicability. Abdollahipour et al. [9] studied the COD of hydraulic fractures using discrete element method (DEM). They proposed an equation to predict COD in HFs in direction of one of the principal stresses. In the current study, a huge amount of numerical modellings has been carried out to study the effects of various parameters (Young's modulus E, Poisson's ratio v, internal pressure P, far-field stresses σ_H and σ_h, crack inclination angle, θ and crack half-length, a) on COD. A multi-parameter regression is performed on theses numerical results in order to propose a relation between COD and the mentioned parameters. The effect of crack

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propagation and well radius on the maximum COD of HFs have been studied too.

EXPERIMENTAL PROCEDURES
Higher Order Displacement Discontinuity Method

Quadratic element displacement discontinuity which is based on analytical integration of cubic collocation shape functions over collinear or straight-line displacement discontinuity elements [10] is used to achieve higher accuracy. Many studies have shown a significant increase in accuracy of the method when higher order elements have been used [11–14]. The far field in-situ stresses are changed to local normal and shear stresses on each element using appropriate stress transition equations. Then the resulted normal or shear local stresses are summed up with the corresponding boundary conditions (if there is any stress boundary) to produce the actual stress state on each element.

Quadratic Displacement Discontinuity Element

The quadratic element displacement discontinuity is based on analytical integration of quadratic collocation shape functions over collinear, straight-line displacement discontinuity elements. Figure 1 shows the quadratic displacement discontinuity distribution, which can be written in a general form as:

$$D_i(\varpi) = N_1(\varpi)D_i^1 + N_2(\varpi)D_i^2 + N_3(\varpi)D_i^3, \ i = x, y \quad (1)$$

Where, D_i^1, D_i^2, and D_i^3 are the quadratic nodal displacement discontinuities, and

$$N_1(\varpi) = \varpi(\varpi - a_2 - a_3) / \left[(a_1 + a_2)(a_1 + 2a_2 + a_3)\right] \text{ and}$$
$$N_2(\varpi) = -(\varpi + a_1 + a_2)(\varpi - a_2 - a_3) / \left[(a_1 + a_2)(a_2 + a_3)\right] \text{ and}$$
$$N_3(\varpi) = \varpi(\varpi + a_1 + a_2) / \left[(a_2 + a_3)(a_1 + 2a_2 + a_3)\right] \quad (2)$$

are the quadratic collocation shape functions. It should be noted that a quadratic element has 3 nodes, which are the centers of the three elements within the batch element.
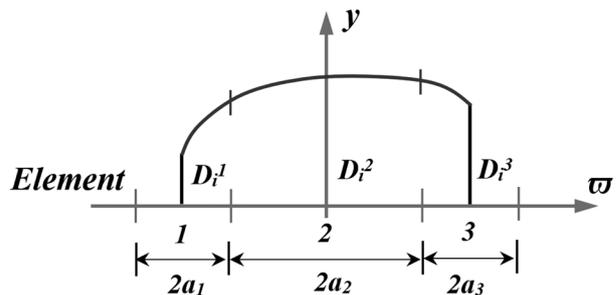

Figure 1: Quadratic collocation for the quadratic element displacement discontinuity.

Crouch has expressed the displacements and stresses for a line crack in an infinite body along the plane y= 0 (x-axis), in terms of single harmonic functions g(x,y) and f(x,y) [15], in which the displacements are

$$u_x = \left[2(1-\nu)f_{,y} - yf_{,xx}\right] + \left[-(1-2\nu)g_{,x} - yg_{,xy}\right] \text{ and}$$
$$u_y = \left[(1-2\nu)f_{,x} - yf_{,xy}\right] + \left[2(1-\nu)g_{,y} - yg_{,yy}\right] \quad (3)$$

and the stresses are:

$$\sigma_{xx} = 2G_s\left[2f_{,xy} + yf_{,xyy}\right] + 2G_s\left[g_{,yy} + yg_{,yyy}\right] \text{ and}$$
$$\sigma_{yy} = 2G_s\left[-yf_{,xyy}\right] + 2G_s\left[g_{,yy} - yg_{,yyy}\right] \text{ and}$$
$$\sigma_{xy} = 2G_s\left[2f_{,yy} + yf_{,yyy}\right] + 2G_s\left[-yg_{,xyy}\right] \quad (4)$$

Where $f_{,x}$, $g_{,x}$, $f_{,y}$, $g_{,y}$, etc. are the partial derivatives of the single harmonic functions f(x,y) and g(x,y) with respect to x and y, in which these potential functions for the quadratic element case can be found from:

$$f(x,y) = \frac{-1}{4\pi(1-\nu)}\sum_{j=1}^{3} D_x^j F_j(I_0, I_1, I_2) \text{ and}$$

$$g(x,y) = \frac{-1}{4\pi(1-\nu)}\sum_{j=1}^{3} D_y^j F_j(I_0, I_1, I_2)$$

$$(5)$$

The common function F_j, is defined as:

$$F_j(I_0, I_1, I_2) =$$

$$\int N_j(\varpi) \ln\left[(x-\varpi) + y^2\right]^{\frac{1}{2}} d\varpi, \quad j=1, \text{ to } 3 \qquad (6)$$

Where the integrals I_0, and I_1 and I_2 are expressed as follows:

$$I_0(x,y) = \int_{-a}^{a} \ln\left[(x-\varpi)^2 + y^2\right]^{\frac{1}{2}} d\varpi$$

$$= y(\varphi_1 - \varphi_2) - (x-a)\ln(r_1) + (x+a)\ln(r_2) - 2a$$

and

$$I_1(x,y) = \int_{-a}^{a} \varpi \ln\left[(x-\varpi)^2 + y^2\right]^{\frac{1}{2}} d\varpi$$

$$= xy(\varphi_1 - \varphi_2) + 0.5(y^2 - x^2 + a^2)\ln\frac{r_1}{r_2} - ax$$

and

$$I_2(x,y) = \int_{-a}^{a} \varpi^2 \ln\left[(x-\varpi)^2 + y^2\right]^{\frac{1}{2}} d\varpi =$$

$$\frac{y}{3}(3x^2 - y^2)(\varphi_1 - \varphi_2) + \frac{1}{3}(3xy^2 - x^3 + a^3) \times$$

$$\ln(r_1) - \frac{1}{3}(3xy^2 - x^3 - a^3)\ln(r_2) - \frac{2a}{3}(x^2 - y^2 + \frac{a^2}{3})$$

$$(7)$$

The terms φ_1, φ_2, r_1 and r_2 in this equation are defined as:

$$\varphi_1 = \arctan(\frac{y}{x-a}), \quad \varphi_2 = \arctan(\frac{y}{x+a}),$$

$$r_1 = \left[(x-a)^2 + y^2\right]^{\frac{1}{2}}, and\ r_2 = \left[(x+a)^2 + y^2\right]^{\frac{1}{2}} \qquad (8)$$

Verification of Higher Order Displacement Discontinuity

The accuracy and applicability of the higher order displacement discontinuity method (using quadratic displacement discontinuity elements) is tested against two well-known problems (i.e. the problems of a central slant crack in an infinite body and a suddenly pressurized crack). The element length has been kept constant and equal to 5cm for all numerical analyses unless told otherwise.

Central Slant Crack in av Infinite Body

Central slant crack problem (see Figure 2) is used to verify the accuracy of the proposed quadratic formulation of the DDM. The slant angle, γ = 60° and a half crack length d =0.5 m are assumed in the analysis. A crack tip element length to half crack length ratio, L/d= 0.1 has been used for the numerical solution of the problem (L is crack tip element length). The analytical solution of the first and second mode stress intensity factors (SIFs), K_I and K_{II}, for the center slant crack problem are given as [16,17].

$$K_I = \sigma\sqrt{\pi d}\sin^2\gamma \text{ and}$$

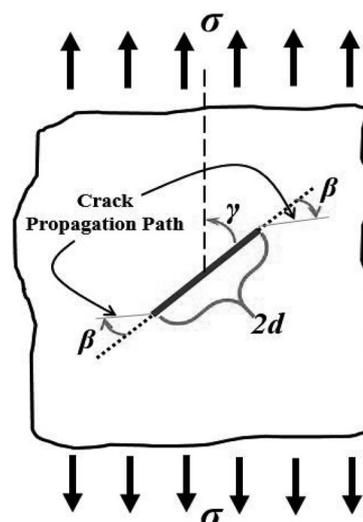$$K_{II} = \sigma\sqrt{\pi d}\sin\gamma\cos\gamma \qquad (9)$$

Figure 2: Central slant crack in an infinite plate.

Figure 3 compares the normalized analytical and numerical SIFs ($K_I/\sigma\sqrt{\pi a}$ and $K_{II}/\sigma\sqrt{\pi a}$) for various number of elements. The best results are achieved for 30 quadratic displacement discontinuity elements (i.e. element length of almost 3 Cm). For the higher number of elements, the crack tip element may loses its significance resulting in lower accuracy of SIF prediction [4]. Using 30 elements the error in estimating SIFs is less than 1%, showing the high accuracy of the proposed method.

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A Suddenly Pressurized Crack

A suddenly pressurized crack in an infinite body is used to verify the proposed quadratic element formulation of the displacement discontinuity method for COD estimation.
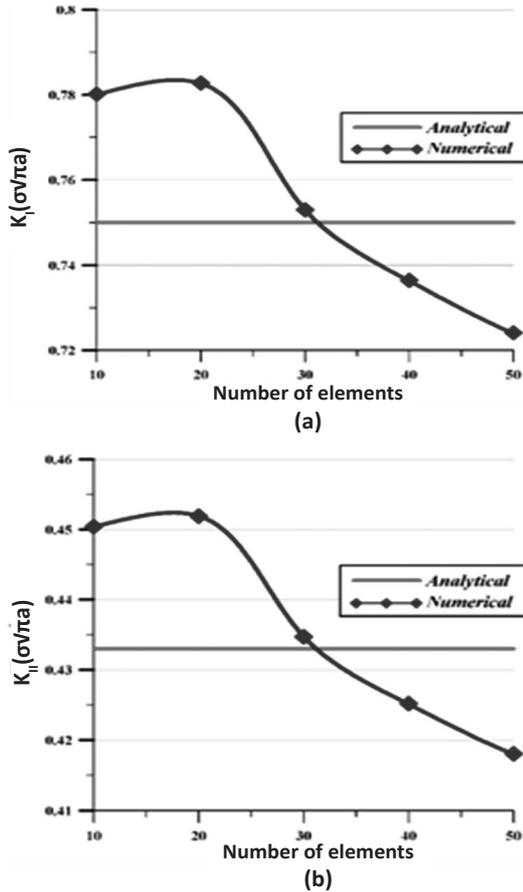

(a)

(b)

Figure 3: Comparison of numerical and analytical results of stress intensity factors a) Normalized K_I b) Normalized K_{II}.

Figure 4 shows the problem of a suddenly pressurized crack. A thin crack with length of 2a is subjected to constant internal pressure P. Sneddon calculated the exact amount of relative normal displacement of the crack surfaces (i.e. crack opening displacement or COD) in an elastic medium [18].

$$COD = \frac{4P(1-v^2)}{E}\sqrt{a^2-x^2} \qquad (10)$$
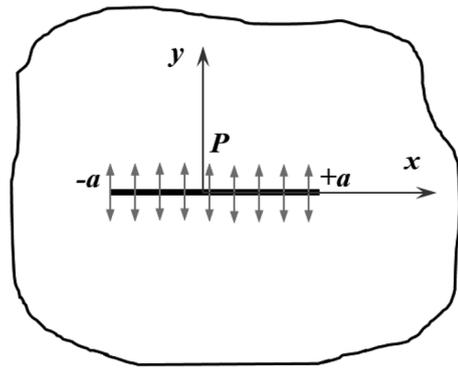
where $-a \leq x \leq a$.

Figure 4: A suddenly pressurized crack problem.

Figure 5 compares the analytical and numerical predictions of COD. As it can be seen the numerical results are in good agreement with the analytical values. The error in COD estimation is less than 1% therefore, the proposed numerical model predicts COD with a significant accuracy.
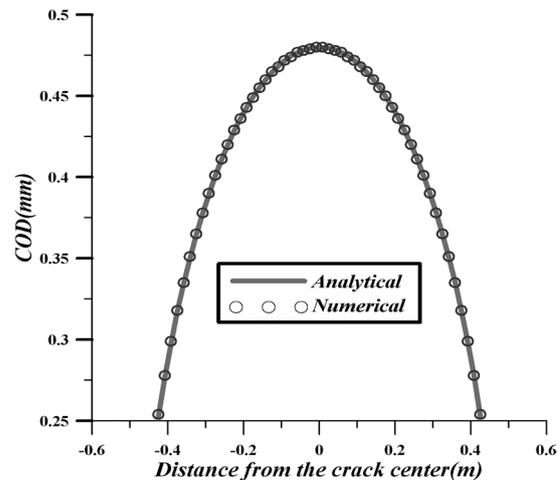

Figure 5: Comparison of the analytical values and numerical prediction of COD for a pressurized crack problem.

Crack Propagation Prediction

The pre-cracked rock-like cylindrical specimens with 60 mm diameters and 120 mm lengths are specially prepared by mixing the Portland Pozzolana cement (PPC), fine sands and water. The mechanical properties of the un-cracked rock-like specimens are obtained by using laboratory tests following ISRM standards. The mechanical

properties used in the present analysis are: Compressive strength, σ_c=28 MPa; Young's modulus, E=15 GPa, Brazilian tensile strength, σ_t=3.81 MPa; Poisson's ratio, ν =0.21.

Some uniaxial compression tests are conducted on the rock-like specimens containing three cracks 1, 2 and 3. These cracks are created by inserting three thin steel shims with 10 mm width and 1 mm thickness in molds (before casting the specimens). The uniaxial compressive stress, σ was uniformly applied and the loading rate was kept at 0.5 MPa/s during the tests.

Figure 6 illustrates the geometry and loading condition of a pre-cracked specimen with three cracks. Cracks 1 and 2 (Iso-path cracks) have constant orientations of α= 60°. The third crack is oriented at different angles with respect to the direction of cracks 1 and 2 i.e. at angles β=0°, 45° and 90° (in a counterclockwise direction) (see Figure 6). Figure 7 demonstrates the experimental and numerical results of the specimens. As it can be seen the numerical results in Figure 7(b) have successfully captured the behavior of the experimental specimens in Figure 7(a) showing the accuracy and applicability of the numerical method in prediction of crack propagation.
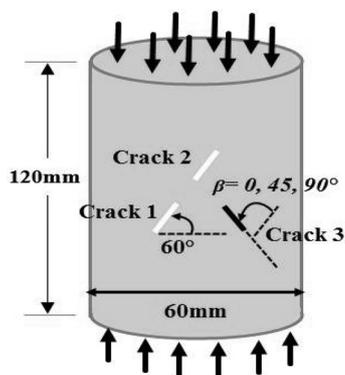

Figure 6: Geometry of the two cracks in a pre-cracked rock-like specimen under uniaxial compression.

β=0°   β=45°   β =90°
(a)

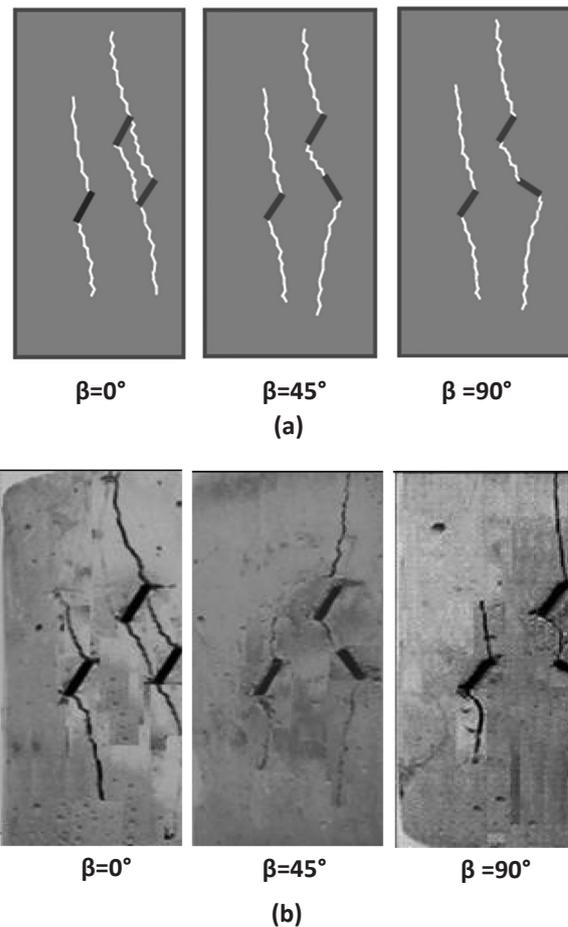β=0°   β=45°   β =90°
(b)

Figure 7: Comparison of crack propagation between experimental (a) and numerical(b) crack propagation for specimens (with three cracks) under uniaxial compression.

RESEARCH METHODOLOGY

A higher order displacement discontinuity method using quadratic element formulation is utilized to model the hydraulic fracture propagation. In this analyses, the rock mass is assumed to be impermeable and a uniform pressure of P is considered in the hydraulic fracture. For the case of far-field stresses, the maximum horizontal stress σ_H is applied in x-axis direction and the minimum horizontal stress σ_h is applied in y-axis direction (Figure 8). A fracture toughness of $K_{IC}= 3MPa\sqrt{m}$ is assumed based on the average value of this parameter in many studies of reservoir conditions [19–24].

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NUMERICAL MODELING

Almost 1500 different conditions have been modeled to fully capture the effect and behavior of each parameter on the COD. Figure 6 shows a schematic view of the model and parameters used in the numerical modeling. Angle θ changes from 0° to 75° with 15° intervals. The other parameters are shown in Table 1. In this numerical simulation, E is Young's modulus and v is the Poisson's ratio of the rock.
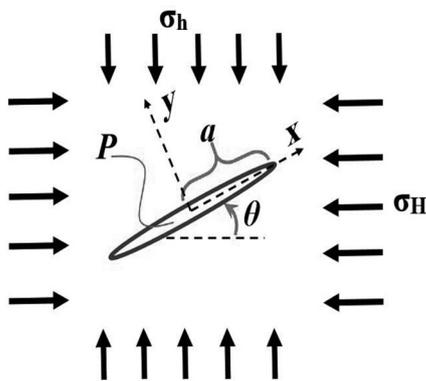

Figure 8: Schematic of models.

Table 1: Various values of parameters used in the numerical modeling.

P/σ_H	σ_h/σ_H	Crack half length(m)	E(GPa)	v
1	0	0.25	30	0.20
1.5	0.35	0.50	40	0.25
2	0.70	0.75	50	0.30

Multivariate Regression

In order to propose a relation to predict COD by taking into account the effect of various parameters, a multivariate non-linear regression analysis is used. One of the classical problems in this statistical analysis is to find a suitable relationship between a response variable (in this case COD) and a set of regression variables (in this case, the parameters; E, v, P/σ_H, σ_h/σ_H, a, x, and θ) under suitable

assumptions on the errors. The usual assumption is that the errors are independent, identically distributed random variables. In many situations the response function is a nonlinear function in both regression variables and the parameters (which is the case in the current model).

A merit function (which measures the agreement between the input data and a regression model) was chosen as follows:

$$\chi^2 = \sum_{i=1}^{N} \left\{ \frac{y_i - y(x_i)}{\sigma_i} \right\}^2 \tag{11}$$

where σ_i is the standard deviation of the ith data point. y_i is the actual (numerical) value of COD, $y(x_i)$ is the predicted COD, x_i are independent variables. The Levenberg-Marquardt method is used for the initial estimation. Eq. (12) shows the best fit on the data. Coefficient of determination, R^2 for this equation is 94.35%. The standard error of the estimate is 4.37×10^{-4} further showing the goodness of the performed fit.

$$COD = 0.214 \frac{(1-v^2)(P-\sigma_h)(1+cos^2\theta)}{2E\sigma_H} \left(\sqrt{a^2 - r^2} \right) \tag{12}$$

where $r = \sqrt{x^2 + y^2}$ is the length along the crack direction, x and y are in crack local coordinates shown in Figure 8. It should be noted that for a horizontal crack (θ=0, y=0) the proposed equation simplifies to an equation very similar to the analytical solution proposed by Sneddon in Eq. (10).

As derived in an experimental study by Gruesbeck and Collins a crack width to average particle diameter of 3 will handle any concentration of sand (proppant) [25]. Equation 12 may be used in early stages of hydraulic fracturing treatment design to make sure the minimum desired width of hydraulic fracture has been obtained for safe passage of

proppant.

A fracture propagation pressure or so called "fracture gradient" of order of 0.7 psi/ft is necessary to initiate the fracture propagation in hydraulic fractures [26].

One of the direct applications of the proposed equation in the paper is determination of dimensionless conductivity of a hydraulic fracturing treatment, leading to FOI, the expected folds of increase of productivity due to a fracture job[26].

RESULTS AND DISCUSSION
Effect of Crack Propagation on Initial Crack Length COD

The effect of crack propagation on COD of the initial crack is also investigated (for the same models of the previous section). The numerical results showed that propagation of HFs increases the maximum COD of the initial crack length. The relation is almost linear in all cases. As shown in Figure 9 for three different conditions, a linear equation is fitted with a high coefficient of determination (almost equal to 1). The x-axis stands for the ratio of new crack length to initial crack length and y-axis represents the maximum COD in the initial crack length. In the present analyses, for set (a) the parameters are σ_h/σ_H=0.7, P/σ_H=1.5, θ=0°, E=30GPa, v=0.20, and a=0.50 m, for set (b) σ_h/σ_H=0.7, P/σ_H=1.0, θ=30°, E=50GPa, v=0.25, and a=0.75 m and for set (c) σ_h/σ_H=0.35, P/σ_H=1.0, θ=60°, E=40GPa, v=0.30, and a=0.25 m. As it can be seen, the value of COD has increased almost linearly for every increment of crack propagation regardless of the mechanical or geometrical properties of the model. It should be also noted that as the angle, θ increases, the effect of crack propagation on COD decreases (comparing the coefficient of x in the fitted equations shown in Figure 9). The maximum effect of crack propagation

on COD is when the HF is in direction of maximum horizontal stress (i.e. θ=0°). This result is in agreement of many experimental and field studies showing perforation and crack propagation in direction of the maximum horizontal stress is the most efficient form of hydraulic fracturing [3,27-29].
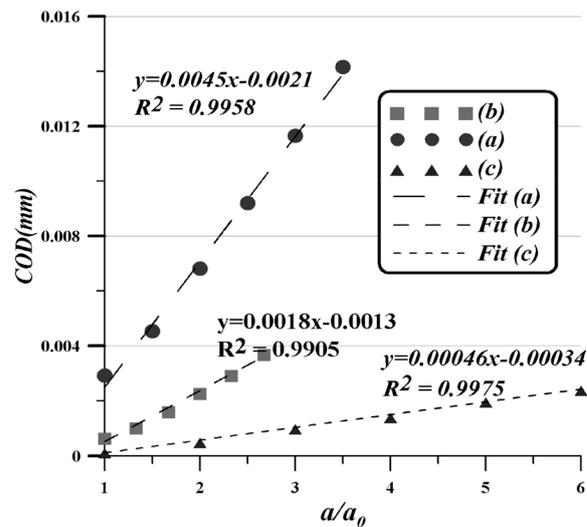

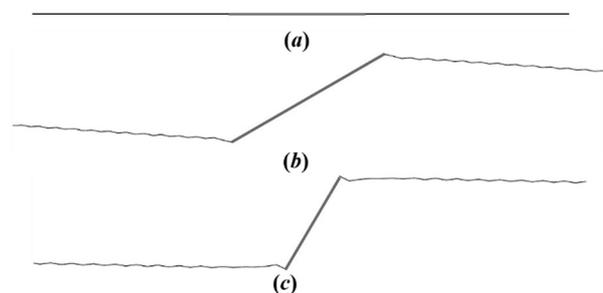**Figure 9: Increase of COD of initial crack with propagation.**

Figure 10: Crack propagation for data sets of Figure 9.

The increase of COD improves the production rate by increasing the volume of hydrocarbon entering the crack and finally reaching the well. Figure 10 shows the propagation of data set (c) in Figure 10. As expected, these results show that the crack propagation is in the direction of maximum horizontal stress. It should be mentioned that angle of initial crack in relation to stress field does affect crack propagation in early steps of the numerical simulation. However, this near crack tip effect is

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not clearly visible in Fig 10 since all cracks reorient themselves in direction of maximum stress as dictated by the theory of LEFM.

Effect of Well Radius on COD

A hydraulic fracture in a well stimulation process starts from the well wall and propagates through the hydrocarbon reservoir. It is interesting to investigate the effect of well size on the maximum obtainable COD. Therefore, in the present analyses the effect of well radius on the COD is also investigated. Five different radiuses are selected. Figure 11 shows a typical model used for analysis of well radius effects on COD. Radius R is changed from 0.1 m to 0.5 m with 0.1 m intervals. The internal pressure has been kept constant and equal to 60 MPa in all models. The maximum COD occurs at the contact of hydraulic fracture and the oil or gas well. The results show that the size of well has the same trend (but as will be explained in following, not the same significant effect) as the previous part. As it is shown in Figure 12 for various inclination angles, the maximum COD changes almost linearly with the well radius. As for the previous section comparing the coefficient of x in the fitted equations (Figure 12), the maximum effect of well radius is when HFs have the least angle with the maximum horizontal stress (i.e. for θ=30° in Figure 12).
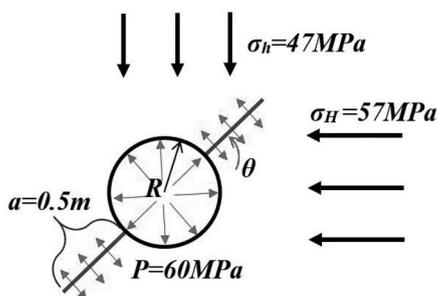

Figure 11: Schematic of the model for analysis of effect of well radius on maximum COD.

As seen in Figure 12 the value of COD has not changed substantially with well radius. This shows that well radius has marginal effect on COD.
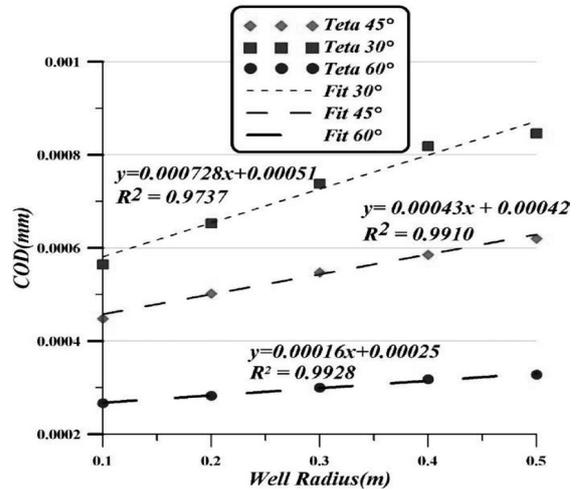

Figure 12: Effect of well radius on maximum COD of hydraulic fractures.

CONCLUSIONS

Crack opening displacement (COD) is an important parameter in fracture mechanics literature andhydraulic fracturing of hydrocarbon reservoirs. The existing analytical solution for COD considers a very simple problem limiting its applicability in the estimation of COD under a certain condition. In this paper, the effect of various parameters on COD was investigated. A higher order displacement discontinuity method was modified to analyze the different conditions. The applicability and accuracy of the model was tested against the two existing analytical solutions. The error in COD was less than 1%, showing the high accuracy of the proposed numerical model. Nearly 1500 models were built and run to capture the effect of all parameters.

An equation was fitted on the resulted predictions of COD of an HF in any arbitrarily condition. The coefficient of determination and standard error of the estimated COD were 94.35% and 4.37×10^{-4} respectively, showing a good agreement between

he fitted equation and numerical results. Since the numerical model was able to predict COD of the analytical problem with less than 1% error, the resulted equation is expected to predict COD of various conditions with an accuracy of the same order (there is a marginal difference between numerical and the proposed equation results which may negligibly increase the error of the proposed equation more than the numerical results). It was also observed that in case of crack propagation and increase of well radius, the COD increases almost linearly regardless of the other parameters in the model.

NOMENCLATURES

COD	: Crack Opening Displacement
HFs	: Hydraulic Fractures
SIFs	: Stress Intensity Factors

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